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$$\int \frac{1}{\sqrt{\sqrt{x}+1}}\, dx$$

Substitution $\boxed{x = z^2}$

<u>Rechnung dx</u>

$$x = [z(x)]^2 \quad | \; '$$

$$1 = z' \cdot 2 \cdot z^1(x) \quad | \cdot$$

$$1 = \frac{dz}{dx} \cdot 2z \quad | \cdot dx$$

$$\boxed{dx = 2z\, dz}$$

$$\int \frac{1}{\sqrt{\sqrt{z}+1}}\, dx$$

$$= \int \frac{1}{\sqrt{\sqrt{z^2}+1}}\, 2z\, dz$$

$$= 2\int \frac{z}{\sqrt{z+1}}\, dz$$

„einfacher", aber immer noch „schwierig"

2. Substitution $\sqrt{z+1} = u(z) \quad |()^2$

$$z+1 = u^2(z)$$

$$\boxed{z = u^2(z)-1}$$

Berechnung von dz

$$z = u^2(z)-1 \quad |\,'$$

$$z = [u(z)]^2$$

$$1 = u' \cdot 2 \cdot u^1(z) - 0 \quad |\, u' = \frac{du}{dz}$$

$$1 = 2\,\frac{du}{dz}\cdot u \quad | \cdot dz$$

$$\boxed{dz = 2u\,du}$$

$$2\int \frac{z}{\sqrt{z+1}}\,dz$$

$$= 2\int \frac{u^2-1}{\cancel{u}}\,2\cancel{u}\,du$$

$$\boxed{= 4\int (u^2-1)\,du}$$ erheblich einfacher!

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$$4\int (u^2-1)\,du$$

$$= 4\cdot\left[\frac{1}{3}u^3-u\right]$$

$$= \frac{4}{3}u^3-\frac{4}{\cancel{3}}u$$

Rücksubst

$u=\sqrt{z+1}$

$u^3=(z+1)\sqrt{z+1}$

$$= \frac{4}{3}\left((z+1)\sqrt{z+1}\right)-\frac{4}{\cancel{3}}\sqrt{z+1}$$

Rücksubst

$z=\sqrt{x}$

$$= \frac{4}{3}\left[(\sqrt{x}+1)\sqrt{\sqrt{x}+1}\right]-\frac{4}{\cancel{3}}\sqrt{\sqrt{x}+1}+C$$

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$$\int (\sin x \cdot \cos x)^3 \, dx$$

$$= \int \sin^3 x \cdot \cos^3 x \, dx$$

Substitution $\sin x = z(x)$

$(\sin^2 + \cos^2 = 1)$

Berechnung von dx

$$\sin x = z(x) \quad \Big|\, ' \qquad z' = \frac{dz}{dx}$$

$$\cos x = \frac{dz}{dx} \quad \Big| \cdot dx$$

$$\boxed{\cos x \, dx = dz}$$

$$\int \sin^3 x \cdot \cos^3 x \, dx$$

$$= \int \sin^3 x \cdot \underbrace{\cos^2 x \cdot \cos x \, dx}$$

$$= \int z^3 \cdot (1 - \sin^2) \cdot dz$$

$$= \int z^3 \cdot (1 - z^2) \, dz$$

$$= \int (z^3 - z^5)\, dz$$ erheblich einfacher

$$= \frac{1}{4} z^4 - \frac{1}{6} z^6$$

Rücksubstitution $z = \sin x$

$$= \frac{1}{4} \sin^4(x) - \frac{1}{6} \sin^6(x) + C$$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

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Lösungen zu
485 Teil 3

Substitution $\boxed{x = \sin(z)}$ '

$$1 = z' \cdot \cos(z)$$

$$1 = \frac{dz}{dx} \cdot \cos(z)$$

$$\boxed{dx = \cos z\, dz}$$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2(z)}{\sqrt{1-\sin^2(z)}} \cos z\, dz$$

$$= \int \frac{\sin^2(z)}{\sqrt{\cos^2(z)}} \cdot \cos z\, dz$$

$$= \int \frac{\sin^2(z)}{\cos(z)} \cdot \cos(z)\, dx$$

$$= \int \sin^2(z)\, dz$$ viel einfacher!

$$\int \underset{u'}{\sin(z)} \cdot \underset{v}{\sin(z)}\,dz = -\cos \cdot \sin - \int -\cos \cdot \cos$$

$$= -\sin \cdot \cos + \int \cos^2$$

$$\overset{!}{=} -\sin \cdot \cos + \int 1 - \sin^2$$

$$= -\sin \cdot \cos + \int 1 - \int \sin^2$$

$$= -\sin \cdot \cos + z - \int \sin^2$$

Zwischenergebnis

$$\int \sin^2 = -\sin\cos + z - \int \sin^2 \quad \Big| + \int \sin^2$$

$$2\int \sin^2 = -\sin\cos + z \quad \Big| :2$$

$$\int \sin^2 = \frac{1}{2}\Big[z - \sin\cos\Big]$$

Rücksubst.

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \sin^2(z)\,dz$$

$$= \frac{1}{2}\Big[z - \sin(z)\cdot\cos(z)\Big]$$

$\sin z = x$

$z = \arcsin(x)$

$$= \frac{1}{2}\cdot\Big[\arcsin(x) - x \cdot \cos z\Big]$$

Weiter mit

$$\sin(z) = x$$

$$\sqrt{1-\cos^2(z)} = x \quad |()^2$$

$$1-\cos^2(z) = x^2 \quad |-1$$

$$-\cos^2(z) = x^2-1 \quad |\cdot(-1)$$

$$\cos^2(z) = 1-x^2 \quad |\sqrt{}$$

$$\cos(z) = \sqrt{1-x^2}$$

$$= \frac{1}{2}\left[\arcsin(x) - x \cdot \sqrt{1-x^2}\right] + C$$

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