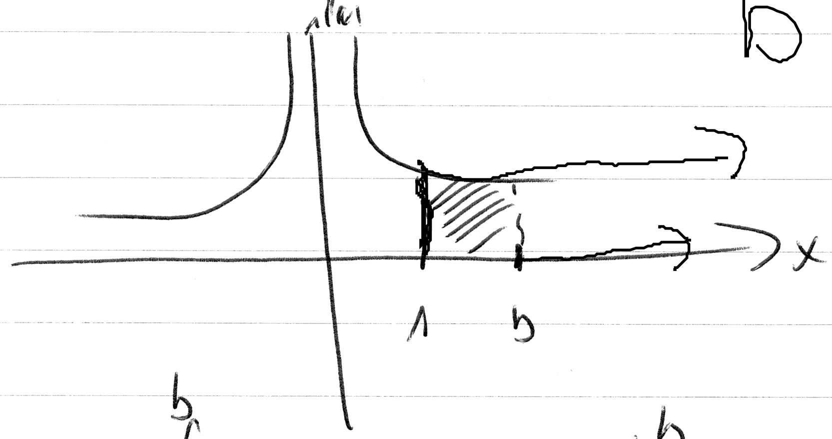
Uneigentliche Integrale

1. Beispiel $\quad f(x) = \frac{1}{x^2}$

$b > 1$



$$A_b = \int_1^b \frac{1}{x^2}\,dx = \int_1^b x^{-2}\,dx = -x^{-1}\Big|_1^b$$

$$= -\frac{1}{x}\Big|_1^b = \left[-\frac{1}{b}\right] - \left[-1\right] = \underline{1 - \frac{1}{b}}$$

Überlegung: Falls $b \to \infty$, ist

$$\lim_{b \to \infty} \left[1 - \frac{1}{b}\right] \overset{!}{=} 1 - 0 = \underline{\underline{1}}$$

„Wert" $\notin \mathbb{R}$

Festlegung $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ heißt „uneigentliches Integral"

und bedeutet

$$\boxed{\int_1^{+\infty} \frac{1}{x^2}\,dx} \overset{!}{=} \lim_{\substack{b \to \infty \\ b > 1}} \int_1^{b} \frac{1}{x^2}\,dx \qquad b \in \mathbb{R}$$

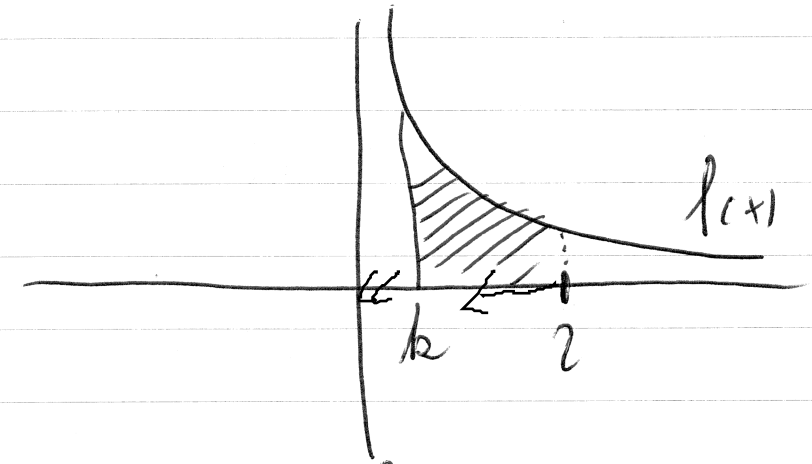
Interpretation $\displaystyle\lim_{b \to \infty} \int_1^{b} \frac{1}{x^2}\,dx \overset{!}{=} 1$

Man ordnet der „unendlich reichenden Fläche" den Wert „1" zu

$\boxed{\text{2. Beispiel}}$ Wir betrachten

$f(x) = \frac{1}{\sqrt{x}}$ $\qquad 0 < k < 2$

$D(f) = \mathbb{R}^{>0}$ $\qquad a = 2$



$$A = \int_k^2 \frac{1}{\sqrt{x}}\,dx = \int_k^2 x^{-\frac{1}{2}}\,dx = \underline{2 \cdot x^{\frac{1}{2}}}\Big|_k^2$$

$$= \left[2 \cdot 2^{1/2}\right] - \left[2 \cdot k^{1/2}\right] = \underline{\underline{2\sqrt{2} - 2\sqrt{k}}}$$

Für $k \to 0$ gilt nun

$$\lim_{k \to 0} \left[2\sqrt{2} - 2\sqrt{k}\right] \stackrel{!}{=} \underline{\underline{2\sqrt{2}}}$$

Für das „uneigentliche Integral"

$\int_0^2 \sqrt{\frac{1}{x}}\,dx$ gilt:

0 nicht aus $D(f)$!!! $D(f) = \mathbb{R}^{>0}$

$$\int_0^2 \frac{1}{\sqrt{x}}\,dx = \lim_{\substack{k \to 0 \\ k > 0 \\ k < 2}} \int_k^2 \frac{1}{\sqrt{x}}\,dx \overset{!}{=} 2\sqrt{2}$$

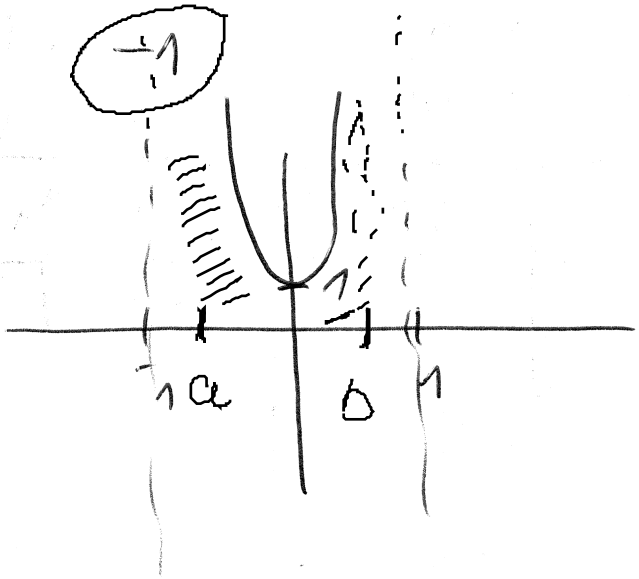
3. Beispiel Wir untersuchen

$$\int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}}\,dx \qquad D(f)=\mathbb{R}^{-1<x<1}$$

$1-x^2>0$

$1>x^2$

$x^2<1$



Offenbar ist

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\substack{a\to -1^+\\ b\to +1^-}} \boxed{\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx} = \pi$$

1. Schritt Bestimmung von

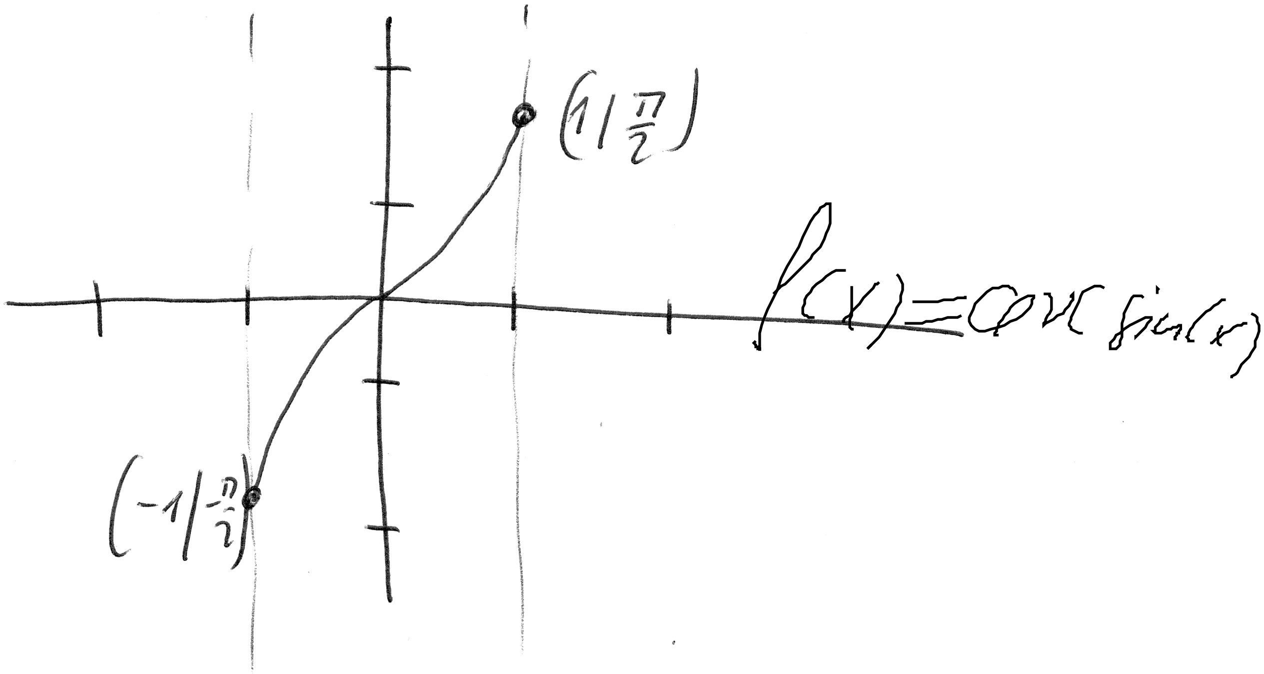
$$\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int_a^b \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \arcsin(x)\Big|_a^b$$

$$= \underline{\underline{\arcsin b - \arcsin a}}$$

<u>Zur Veranschaulichung</u>



$$\underbrace{\lim_{\substack{a \to +1^+ \\ b \to +1^-}} \int_a^b \frac{1}{\sqrt{1-x^2}}\,dx}$$

$$= \underbrace{\lim_{\substack{a \to -1^+ \\ b \to +1^-}} \left[\arcsin b - \arcsin a\right]}$$

$$= \lim_{b \to +1^-} \left[\arcsin(b) - \underbrace{\arcsin(-1)}_{-\frac{\pi}{2}}\right]$$

$$= \underbrace{\arcsin(+1)} + \frac{\pi}{2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \underline{\underline{\pi}}$$